Multiplication of polynomials

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e.g. Multiply $(x+2)$ by (x^2+3x-1).

Method 1: $(x+2)(x^2+3x-1)$ Use Distributive law

$$= (x+2)x^2 + (x+2)\cdot 3x - (x+2)$$

$$= x^3 + 2x^2 + 3x^2 + 6x - x - 2$$

$$= x^3 + 5x^2 + 5x - 2$$

Method 2: Pick combinations that give:

$(x+2)(x^2+3x-1)$

$x^3 \rightarrow x^3$

$(x+2)(x^2+3x-1)$

$x^2 \rightarrow 3x^2 + 2x^2 = 5x^2$

$(x+2)(x^2+3x-1)$

$x \rightarrow -x + 6x = 5x$

$(x+2)(x^2+3x-1)$

constant term $\rightarrow -2$

$$\therefore (x+2)(x^2+3x-1) = x^3 + 5x^2 + 5x - 2$$

Division of polynomials

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e.g. Divide $x^3 + 5x^2 + 5x - 2$ by $x+2$.

Long division. e.g. Divide 1428 by 12.

$$
\begin{array}{rl}
 & 119 \leftarrow \text{quotient} \\
12 \,\big)\!\!\! & \overline{1429} \\
 & -\underline{12} \\
 & 22 \\
 & -\underline{12} \\
 & 109 \\
 & -\underline{108} \\
 & 1 \leftarrow \text{remainder}
\end{array}
$$

$x^3 + 5x^2 + 5x - 2$

- Treat each term like a place of a number.

- Treat each coefficient like a digit.

$$
\begin{array}{rl}
 & \phantom{x^3 + {}} x^2 + 3x - 1 \leftarrow \text{quotient} \\
x+2 \,\big)\!\!\! & \overline{x^3 + 5x^2 + 5x + 6} \\
 & -\underline{(x^3 + 2x^2)} \\
 & \phantom{x^3 + {}} 3x^2 + 5x \\
 & -\underline{(3x^2 + 6x)} \\
 & \phantom{x^3 + 5x^2 + {}} -x + 6 \\
 & -\underline{(-x - 2)} \\
 & \phantom{x^3 + 5x^2 + 5x + {}} 8 \leftarrow \text{remainder}
\end{array}
$$

Remainder Theorem

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e.g. Divide 1429 by $\underline{12}$. Answer: quotient is 119, remainder is 1.

Related by: $1429 = 119 \times \underline{12} + 1$

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e.g. Divide $f(x) = x^3 + 5x^2 + 5x + 6$ by $x+2$.

Answer: Quotient is $x^2 + 3x - 1$
Remainder is 4.

Related by: $f(x) = (\underbrace{x^2 + 3x - 1}_{\text{quotient}})(x+2) + \underbrace{8}_{\text{remainder}}$

Notice something - if $x+2$ is 0, (so $x=-2$)
then $f(x) = f(-2) =$ remainder.

<u>Remainder Theorem</u>:

Let $f(x)$ be a polynomial. (Like $x^3 + 5x^2 + 5x + 6$.)
Divide it by $x-a$. (Like $x+2$)
Then $f(a) =$ remainder. (Like $f(-2) = 8$)

Explanation.

$f(x) =$ quotient $\times (x-a)$ + remainder

$\therefore f(a) =$ quotient $\times 0$ + remainder

$=$ remainder.

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e.g. Find the remainder when $f(x) = x^3 + 5x^2 + 5x + 6$ is divided by $x+2$.

Ans. Don't need to divide. Just calculate $f(-2) =$ __ .

Factor Theorem

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e.g. $f(x) = x^3 + 5x^2 + 5x - 2 = (x+2)(x^2 + 3x - 1)$

$(x+2)$ is a factor. So $f(-2) = 0$.

$\Rightarrow$ If we know that $f(-2) = 0$, then $(x+2)$ must be a factor.' – even if we don't know what $f(x)$ is.

Factor theorem.

Let $f(x)$ be a polynomial. (Like $x^3 + 5x^2 + 5x - 2$.)
Suppose $f(a) = 0$. (Like $f(-2) = 0$)
Then $x - a$ must be a factor. (Like $x+2$).

Explanation:
· Remainder theorem $\Rightarrow$ if we divide $f(x)$ by $x - a$, then $f(a) =$ remainder.
So $f(a) = 0$ means remainder $= 0$.

$\therefore f(x) = \text{Quotient} \times (x - a)$
So $x - a$ is a factor.

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e.g. $f(x) = x^3 + 5x^2 + 5x - 2$. Find $f(-2)$. Then find a factor of $f(x)$.

Ans. $f(-2) = \ldots = 0$.
Factor theorem $\Rightarrow$ $(x+2)$ is a factor.

Factorisation of Polynomials

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e.g. Factorise $f(x) = x^3 + 5x^2 + 5x - 2$.

Trial and error: - Calculate $f(x)$ for $x = 0, 1, -1, 2, -2$.
- Stop when $f(x) = 0$.*

Find

$f(-2) = \ldots = 0$.

Factor Theorem $\Rightarrow$ $(x+2)$ is factor.

To factorise:

Method 1: Divide $f(x)$ by $(x+2)$ to get other factor.

Method 2: Let $x^3 + 5x^2 + 5x - 2 = (x+2)(ax^2 + bx + c)$.

Compare coefficients:

	Left		Right
x^3	1	=	a
x^2	5	=	$b + 2a$
x	5	=	$c + 2b$
constant	-2	=	$2c$

Solving $\rightarrow$ $a = 1$, $b = 3$, $c = -1$

$\therefore$ $x^3 + 5x^2 + 5x - 2 = (x+2)(x^2 + 3x - 1)$.

factorise if possible

* Check again. If sure, then try $x = -3, 3$. Still $\neq 0$? Next question.

Two Identities

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(I) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

(II) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

Proof: (I) Right hand side

$$= (a+b)a^2 - (a+b)ab + (a+b)b^2$$

$$= a^3 + ba^2 - a^2b - ab^2 + ab^2 + b^3$$

$$= a^3 \qquad\qquad\qquad + b^3$$

$=$ left hand side.

Proof: (II) Right hand side

$$= (a-b)a^2 + (a-b)ab + (a-b)b^2$$

$$= a^3 - ba^2 + a^2b - ab^2 + ab^2 - b^3$$

$$= a^3 \qquad\qquad\qquad - b^3$$

$=$ left hand side.

Cubic Equations

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e.g. Solve $f(x) = x^3 + 5x^2 + 5x - 2. = 0$

Try $x = 0, 1, -1, 2, -2, \ldots$ find $f(-2) = 0$

Factor theorem $\Rightarrow$ $x+2$ is factor.

Factorise: Method 1 - divide $f(x)$ by $x+2$.

Method 2. Compare coefficients

$$x^3 + 5x^2 + 5x - 2 = (x+2)(ax^2 + bx + c).$$

Get $x^3 + 5x^2 + 5x - 2 = (x+2)(x^2 + 3x - 1)$.

To solve $(x+2)(x^2+3x-1) = 0$

Get $x + 2 = 0$ or $x^2 + 3x - 1 = 0$

$x = -2$

If cannot factorise easily, check $b^2 - 4ac$.

If ≥ 0,

find $x = \dfrac{-3 \pm \sqrt{3^2 - 4(-1)}}{2}$

$= \dfrac{-3 \pm \sqrt{13}}{2}$

Answers: $x = -2$, $\dfrac{-3 + \sqrt{13}}{2}$ or $\dfrac{-3 - \sqrt{13}}{2}$

Partial Fractions 1

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e.g. Write $\dfrac{1}{(x-1)(2x+3)}$ in the form $\dfrac{A}{x-1} + \dfrac{B}{2x+3}$.

Method:
Let $\dfrac{1}{(x-1)(2x+3)} = \dfrac{A}{x-1} + \dfrac{B}{2x+3}$.

Get rid of denominators: $\rightarrow$ Multiply by $(x-1)(2x+3)$

get $1 = A(2x+3) + B(x-1)$.

Choose x to set each bracket to 0:

$x = 1$, $\quad 1 = A(2+3) + B(0) \Rightarrow A = \frac{1}{5}$

$x = -\frac{3}{2}$, $\quad 1 = A(0) + B\left(-\frac{3}{2} - 1\right) \Rightarrow B = -\frac{2}{5}$

Answer: $\dfrac{1}{(x-1)(2x+3)} = \frac{1}{5}\left(\dfrac{1}{x-1}\right) - \frac{2}{5}\left(\dfrac{1}{2x+3}\right)$.

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e.g. Write $\dfrac{x+2}{(x-1)(2x+3)}$ is the form $\dfrac{A}{x-1} + \dfrac{B}{2x+3}$.

Method: Similar $\Rightarrow$

$$x+2 = A(2x+3) + B(x-1)$$

Set $x = 1$: $\quad 1+2 = A(2+3) + B(0)$

$x = -\frac{3}{2}$: $\quad -\frac{3}{2} + 2 = A(0) + B\left(-\frac{3}{2} - 1\right)$ ....

Partial Fractions 2

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e.g. Write $\frac{1}{(x-1)^2(2x+3)}$ in the form $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x+3}$.

Method: Let $\frac{1}{(x-1)^2(2x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x+3}$.

To get rid of denominators, multiply by $(x-1)^2(2x+3)$:

$$1 = A(x-1)(2x+3) + B(2x+3) + C(x-1)^2$$

Try x values to set each bracket to 0 :

$x = 1 \quad \rightarrow \quad 1 = 0 + B(2+3) + 0$

$x = -\frac{3}{2} \quad \rightarrow \quad 1 = 0 + 0 + C(-\frac{3}{2} - 1)^2$

Need to try 3 values $\because$ 3 unknowns. Something simple like

$x = 0 \quad \rightarrow \quad 1 = A(-1)(3) + B(3) + C(-1)^2$

Solving, find $B = \frac{1}{5}$, $C = \frac{4}{25}$, $A =$ ___.

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e.g. Write $\frac{x^2+2x+5}{(x-1)^2(2x+3)}$ in the form $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x+3}$.

Method: like above.

Partial Fractions 3

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eg. Write $\dfrac{1}{(2x+3)(x^2+4)}$ in the form $\dfrac{A}{2x+3} + \dfrac{Bx+C}{x^2+4}$.

Method: Let $\dfrac{1}{(2x+3)(x^2+4)} = \dfrac{A}{2x+3} + \dfrac{Bx+C}{x^2+4}$.

Multiply by denominators: $1 = A(x^2+4) + (Bx+C)(2x+3)$

Pick x so bracket $(2x+3)$ is zero $\rightarrow$
$x = -\frac{3}{2}$, $\quad 1 = A\left(\left(\frac{3}{2}\right)^2+4\right) + 0$

$A = \frac{4}{25}$

Try $x = 0 \rightarrow \quad 1 = A(4) + C(3)$

$C = \frac{3}{25}$

$x = 1 \rightarrow \quad 1 = A(1+4) + (B+C)(2+3)$

$B = -\frac{2}{25}$

Answer. $\dfrac{1}{(2x+3)(x^2+4)} = \dfrac{1}{25}\left(\dfrac{4}{2x+3}\right) + \dfrac{1}{25}\left(\dfrac{-2x+3}{x^2+4}\right)$

Summary

$$\frac{ex+f}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$ — separate out factors

$$\frac{ex^2+fx+g}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$$

include lower powers

$$\frac{ex^2+fx+g}{(ax+b)(x^2+c^2)} = \frac{A}{ax+b} + \frac{Bx+C}{x^2+c^2}$$ — lower degree (power) than denominator

Problem 1

Dr. K.M. Hock

2013 P2 Q3 The function $f(x) = x^3 + ax + b$, where a and b are constants, is exactly divisible by $x+3$. Given that $f(x)$ leaves a remainder of 56 when divided by $x-4$,

(i) Find the value of a and b.

(ii) determine the number of real roots of the equation $f(x) = 0$.

Solution. (i) $x+3$ is factor $\Rightarrow$ $f(-3) = 0$ — (1)

Remainder theorem $\Rightarrow$ $f(4) = 56$ — (2)

(1) $\rightarrow$ $(-3)^3 + a(-3) + b = 0$ $\Rightarrow$ $-27 - 3a + b = 0$ — (3)

(2) $\rightarrow$ $4^3 + a(4) + b = 0$ $\Rightarrow$ $64 + 4a + b = 0$ — (4)

(4) − (3) : $91 + 7a = 0$ $\Rightarrow$ $a = -13$

$\rightarrow$ (3) : $-27 - 3(13) + b$ $\Rightarrow$ $b = 66$

(ii) $\therefore$ $f(x) = x^3 - 13x + 66$

Let $x^3 - 13x + 66 = \underset{\text{factor}}{(x+3)}(cx^2 + dx + e)$

Compare coefficents of : x^3 $\quad 1 = c$

x^2 $\quad 0 = d + 3c$ $\Rightarrow$ $d = -3$

x

constant term $\quad 66 = 3e$ $\Rightarrow$ $e = 22$

$\therefore$ $f(x) = (x+3)(x^2 - 3x + 22) = 0$

$x = -3$ $\qquad$ $x^2 - 3x + 22 = 0$

$b^2 - 4ac = (-3)^2 - 4(22) < 0$

One real root only.

Problem 2

Dr. K.M. Hock

2013 P1 Q3 Express $\frac{7x+2}{(x^2+4)(x-2)}$ in partial fractions.

Solution. Let $\frac{7x+2}{(x^2+4)(x-2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$

$$7x+2 = A(x^2+4) + (Bx+C)(x-2)$$

$x=2 \Rightarrow \quad 14+2 = A(4+4) + 0 \quad \Rightarrow A=2$

$x=0 \Rightarrow \quad 2 = 4A + C(-2) \quad \Rightarrow C=3$

$x=1 \Rightarrow \quad 9 = 5A + (B+C)(-1)$

$$9 = 5(2) - (B+3)$$

$$B = -2$$

$$\therefore \quad \frac{7x+2}{(x^2+4)(x-2)} = \frac{2}{x-2} + \frac{-2x+3}{x^2+4}$$

Check: $\text{RHS} = \frac{2(x^2+4) + (-2x+3)(x-2)}{(x^2+4)(x-2)}$

$$= \frac{2x^2+8 + (-2x^2+4x+3x-6)}{(x^2+4)(x-2)}$$

$$= \frac{7x+2}{(x^2+4)(x-2)} = \text{LHS}$$